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The Projection and Utilization of Neutrosophic Algebra for Decision-Making Processes

Joseph Oluwakorede Campbell[1], Adebisi Sunday Adesina[1,*]

[1] Department of Mathematics, University of Lagos, Akoka, Nigeria; koredecampbell1@gmail.com; adesinasunday@yahoo.com.

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Abstract

Neutrosophic algebra is an emerging field that extends classical algebra by incorporating the concept of neutrosophy, which deals with the study of indeterminacy. This work projects the theoretical foundations of Neutrosophic algebra and its applications to decision-making in areas of importance. By leveraging several Neutrosophic principles, which introduces the notion of truth, falsehood, and indeterminacy simultaneously, Neutrosophic algebra offers a robust framework for addressing problems in various domains where uncertainty, ambiguity, and incomplete information prevail.

Keywords: Neutrosophic algebra, Decision-making processes, Indeterminacy, Fuzzy set and logic, Uncertainties, Inconsistencies.

1 | Introduction

Neutrosophic algebra represents an innovative and evolving branch of mathematics that extends classical algebraic structures to handle indeterminate, vague, or contradictory information. This project aims to contribute to the ongoing development of Neutrosophic algebra.

Neutrosophic algebra and the study of uncertainty has evolved over the year. Tracking all the way back to Classical philosophy, to logic to the evolution of set then fuzzy set and logic all the way to Neutrosophy and Neutrosophic algebra.

Classical Philosophy emerged in ancient Greece, following a procession from what are known as the Presocratics; to the three great philosophers, Socrates (470–399 BCE), Plato (c. 428–347 BCE), and Aristotle (384–322 BCE); and then to later schools of thought, including the Epicureans and Stoics. As is the case with all ancient societies, knowledge of these thinkers limited by the documentation that has survived. Socrates, for example, wrote down nothing. Rather, Plato wrote dialogues featuring his mentor Socrates engaged in

Corresponding Author: adesinasunday@yahoo.com
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philosophical debate with various individuals in Athens, some of them his fellow citizens and other prominent visitors to the city. Plato sought to distinguish philosophers, seekers of truth, from Sophists, whom he regarded as seeking wealth and fame and peddling in fallacious arguments.

logic in a narrow sense is equivalent to deductive logic. By definition[1], such reasoning cannot produce any information (In the form of a conclusion) that is not already contained in the premises. In a wider sense, which is close to ordinary usage, logic also includes the study of inferences that may produce conclusions[2] that contain genuinely new information. Such inferences are called ampliative or inductive[3], and their formal study is known as inductive logic. They were illustrated by the inferences drawn by clever detectives, such as the fictional Sherlock Holmes[4]. There was a medieval[5] tradition according to which the Greek[6] philosopher Parmenides[7] (5th century BCE) invented logic[8] while living on a rock in Egypt. Other authors too contributed to a growing Greek interest in inference[9] and proof. Early rhetoricians and Sophists[10]—e.g., Gorgias[11], Hippias[12], Prodicus[13], and Protagoras[14] (All 5th century BCE)—cultivated the art of defending or attacking a thesis by means of argument. Socrates[15] (c. 470–399 BCE) said to have attended Prodicus's lectures. Like Prodicus, he pursued the definitions of things, particularly in the realm of ethics[16] and values. Plato continued the work begun by the Sophists[17] and by Socrates. In the Sophist, he distinguished affirmation from negation and made the important distinction between verbs and names (Including both nouns and adjectives). He remarked that a complete statement (Logos) cannot consist of either a name or a verb alone but requires at least one of each. This observation indicates that the analysis of language[18] had developed to the point of investigating the internal structures of statements, in addition to the relations of statements as a whole to one another. Plato's pupil Aristotle[19] (384–322 BCE) would raise this new development to a high art. The systematic study of logic seems to have been undertaken first by Aristotle[20]. Although Plato used dialectic[21] as both a method of reasoning and a means of philosophical training, Aristotle established a system of rules and strategies for such reasoning.

Set theory, as a separate mathematical discipline, begins in the work of Georg Cantor. One might say that set theory was born in late 1873, when he made the amazing discovery that the linear continuum, that is, the real line, is not countable, meaning that its points cannot be counted using the natural numbers. In classical set theory[22], the membership of elements in a set is assessed in binary terms according to a bivalent condition[23], an element either belongs or does not belong to the set.

Fuzzy sets (a.k.a. uncertain sets) are sets[24] whose elements[25] have degrees of membership. Fuzzy sets were introduced independently by Zadeh [1] in 1965 as an extension of the classical notion of set. At the same time, Salii [2] defined a more general kind of structure called an L-relation[26], which he studied in an abstract algebraic[27] context. Fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function[28] valued in the real[29] unit interval [0, 1]. Fuzzy sets

[1] https://www.britannica.com/topic/definition
[2] https://www.britannica.com/dictionary/conclusions
[3] https://www.britannica.com/topic/induction-reason
[4] https://www.britannica.com/topic/Sherlock-Holmes
[5] https://www.merriam-webster.com/dictionary/medieval
[6] https://www.britannica.com/topic/Greek-philosophy
[7] https://www.britannica.com/biography/Parmenides-Greek-philosopher
[8] https://www.britannica.com/topic/logic
[9] https://www.britannica.com/topic/inference-reason
[10] https://www.britannica.com/topic/Sophist-philosophy
[11] https://www.britannica.com/biography/Gorgias-of-Leontini
[12] https://www.britannica.com/biography/Hippias-of-Elis
[13] https://www.britannica.com/biography/Prodicus
[14] https://www.britannica.com/biography/Protagoras-Greek-philosopher
[15] https://www.britannica.com/biography/Socrates
[16] https://www.merriam-webster.com/dictionary/ethics
[17] https://www.britannica.com/topic/Sophist-by-Plato
[18] https://www.britannica.com/topic/language
[19] https://www.britannica.com/biography/Aristotle
[20] https://www.britannica.com/biography/Aristotle
[21] https://www.britannica.com/topic/dialectic-logic
[22] https://en.wikipedia.org/wiki/Set_theory
[23] https://en.wikipedia.org/wiki/Principle_of_bivalence
[24] https://en.wikipedia.org/wiki/Set_(mathematics)
[25] https://en.wikipedia.org/wiki/Element_(mathematics)
[26] https://en.wikipedia.org/w/index.php?title=L-relation&action=edit&redlink=1
[27] https://en.wikipedia.org/wiki/Abstract_algebra
[28] https://en.wikipedia.org/wiki/Membership_function_(mathematics)
[29] https://en.wikipedia.org/wiki/Real_number

generalize classical sets, since the indicator functions[1] (Aka characteristic functions) of classical sets are special cases of the membership functions of fuzzy sets, if the latter only takes values 0 or 1. In fuzzy set theory, classical bivalent sets are usually called crisp sets. The fuzzy set theory can be used in a wide range of domains in which information is incomplete or imprecise, such as bioinformatics[2].

In 1995, experimental writer and innovative painter FlorentinSmarandache wrote a manuscript, the treated subject was of philosophy -Revealing paradoxes- and logics. He had generalized the fuzzy logic, and introduced two new concepts: 1) “Neutrosophy” –Study of neutralities as an extension of dialectics, and 2) its derivative “Neutrosophic”, such as “Neutrosophic logic”, “Neutrosophic set”, “Neutrosophic probability”, and “Neutrosophic statistics” and thus opening new ways of research in four fields: Philosophy, logics, Set Theory, and probability/statistics. It was known to me his setting up in 1980’s of a new literary and artistic avant-garde movement that he called “Paradoxism”, because I received some books and papers dealing with it in order to review them for the German journal “ZentralblattfrMathematik”. He made an inspired connection between literature/arts and science, philosophy. We started a long correspondence with questions and answers. Because paradoxism supposes multiple value sentences and procedures in creation, antisense and non-sense, paradoxes and contradictions, and it is tight with Neutrosophic logic.

Neutrosophy is a philosophical system that deals with the concept of neutralities and their interactions. FlorentinSmarandache, a Romanian mathematician, poet, and researcher introduced it in the 1990s. Neutrosophy is part of a broader family of philosophies that includes paraconsistent logic and dialetheism, all of which explore the nature of contradictions and paradoxes. The term "Neutrosophy" is derived from the Greek words "Neutro" (Neutral or middle), "Sophy" (Wisdom), and "Sophia" (Knowledge), representing a philosophy of neutrality and embracing the indeterminacy inherent in various aspects of human understanding and existence. In a traditional sense, philosophical systems often categorize phenomena as true or false, good or bad, right or wrong. Neutrosophy, however, acknowledges the presence of indeterminacy, incompleteness, and inconsistency in human perception and understanding. It introduces the notion of a "Neutral state", recognizing that not all phenomena can be definitively classified as purely true or false.

The key elements of neutrosophy include:

I. Truth (T): This represents the degree to which a statement or concept is true. It is a measure of the veracity or correctness of the information.

II. Indeterminacy (I): This represents the degree to which a statement or concept is indeterminate or uncertain. It reflects the lack of clear distinction between true and false.

III. Falsehood (F): This represents the degree to which a statement or concept is false. It indicates the extent to which the information is incorrect or contradicts reality.

Neutrosophic algebra is a branch of mathematics that extends traditional algebraic structures to handle indeterminacy, inconsistency, and incomplete information using the Neutrosophic framework. Some key aspects of Neutrosophic algebra:

I. Neutrosophic numbers.

II. Neutrosophic operations.

III. Neutrosophic matrices and linear algebra.

IV. Neutrosophic set theory.

V. Neutrosophic logic.

VI. Neutrosophic probability theory.

VII. Neutrosophic measure.

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[1] https://en.wikipedia.org/wiki/Indicator_function

[2] https://en.wikipedia.org/wiki/Bioinformatics

VIII. Neutrosophic group, pseudo group and semigroup.

IX. Neutrosophic field.

X. Neutrosophic ring e. t. c.

Neutrosophic algebra represents an innovative and evolving branch of mathematics that extends classical algebraic structures to handle indeterminate, vague, or contradictory information. The existing body of Neutrosophic algebra is rich, but there remain challenges and unexplored areas that need attention. This project seeks to address the following key issues:

I. Refinement and expansion of Neutrosophic algebraic structures.

II. Development of efficient computational methods for Neutrosophic algebraic operations.

III. Exploration of real-world applications of Neutrosophic algebra in diverse fields.

In 1995, FlorentinSmarandache, a Romanian mathematician, poet, experimental writer,innovative painter and researcher,made most innovations relating to Neutrosophic algebra. He first published a book called "Neutrosophy/Neutrosophic probability, set and logic", introducing Neutrosophy, a philosophy theory. It serves as the fundamental reference for the field. This was because of his explorations in mathematics, philosophy, and his desire to address paradoxes and uncertainties within formal systems due to the limitations of classical logic and set theory. Smarandache [3] took a set forward by releasing "Neutrosophic set–a generalization of the intuitionistic fuzzy set". This paper introduces Neutrosophic sets as a generalization of the intuitionistic fuzzy sets, providing a formal basis for Neutrosophic algebra. "A unifying field in logics: Neutrosophic logic" was released by Smarandache [4]., expanding on Neutrosophy, it explores the development of Neutrosophic logic, its connection to algebraic structures and its application. In 2003, "Neutrosophic Probability" by Smarandache explores Neutrosophic probability theory, providing a foundation for handling uncertainty in probability calculations [5]. Kandasamy and Smarandache [6] published a paper "Neutrosophic rings" investigating Neutrosophic rings and its properties, contributing to the development of algebraic structures within the Neutrosophic framework."Introduction to Neutrosophic algebra" Published by Smarandache [6]. This book introduces neutrosophic algebra, covering basic concepts, operations, and applications. Years later, Chen [7] wrote a paper "NeutrosophicSets and their applications". Discussing the application of Neutrosophic sets in decision-making and provides examples of real-world applications. Smarandache and Dezert [8] then collaborated on "Advances and applications in Neutrosophic logic and Neutrosophic Set Theory". This is an edited volume, co-edited by Smarandache and Dezert; it presents advances and applications in Neutrosophic logic and set theory. In 2014, "A novel method for single-valued Neutrosophic multi-criteria Decision Making (DM) with incomplete weight information" Published by Zhang and Wu [9], integrates Neutrosophic algebra into decision-making processes, addressing the challenges of incomplete information. Abdel-Basset et al. [10] in 2018 on the paper "A hybrid approach of Neutrosophic sets and DEMATEL method for developing supplier selection criteria" talking about the application of Neutrosophic algebra in feature selection algorithms, emphasizing computational efficiency. Abdel-Basset et al. [11] then published "A group DM framework based on neutrosophic Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) approach for smart medical device selection" showing how Neutrosophic algebra can be applied to medical decision-making using the TOPSIS method. "Neutrosophic soft rough matrices", investigates the integration of Neutrosophic algebra with other mathematical structures, exploring the possibilities of hybrid approaches. In the same year, EmanAboElHamd, Hamed M. Shamma, Mohamed Saleh and Ihab El-Khodary released "Neutrosophic logic Theory and Applications", diving deep into Neutrosophic logic and its applications in various fields. Smarandache and Pramanik [12] published " New trends in neutrosophic theory and applications" A review paper that summarizes recent research trends in Neutrosophic algebra and outlines potential future directions.

2 | Methodology

In Neutrosophic algebra, the basic algebraic operations such as addition, subtraction, multiplication, and division are generalized to accommodate Neutrosophic elements. Neutrosophic elements are entities that have three components: The Truth component (T), the Indeterminacy component (I), and the Falsehood component (F). Each component represents the degree of truth, indeterminacy, or falsehood associated with the element.

The Neutrosophic algebraic operations are defined in a way that takes into account the Neutrosophic components. For example, we define two Neutrosophic numbers A and B as follows:

$$\mathrm{A} = (\mathrm{T_A}, \mathrm{I_A}, \mathrm{F_A}).$$

$$\mathrm{B} = (\mathrm{T_B}, \mathrm{I_B}, \mathrm{F_B}).$$

Then, the Neutrosophic addition of two Neutrosophic numbers (A, B) is given by

$$\mathrm{A} \oplus \mathrm{B} = (\mathrm{T_A} + \mathrm{T_B} - \mathrm{I_A} \cdot \mathrm{F_B}, \mathrm{I_A} \cdot \mathrm{I_B}, \mathrm{F_A} + \mathrm{F_B} - \mathrm{I_A} \cdot \mathrm{T_B}),$$

where $\mathrm{T_A}$, $\mathrm{I_A}$, and $\mathrm{F_A}$ are the truth, indeterminacy, and falsehood components of the Neutrosophic number A, respectively. And $\mathrm{T_B}$, $\mathrm{I_B}$, and $\mathrm{F_B}$ are the truth, indeterminacy, and falsehood components of the Neutrosophic number B, respectively.

3.1 | Neutrosophic Numbers

A Neutrosophic number is represented in the form (a, b, c), where 'a' is the determinate component, 'b' is the indeterminate component, and 'c' is the neutral or imaginary component. Each of these components can be a real or complex number.

I. Determinate component ('a'): This part represents the precise, known, or certain information associated with the Neutrosophic number.

II. Indeterminate component ('b'): This part represents the degree of indeterminacy, uncertainty, or ambiguity associated with the Neutrosophic number. It reflects the range of possible values or the extent to which the information is not well-defined.

III. Neutral component ('c'): This part represents the neutral or imaginary information associated with the Neutrosophic number. It can be related to hypothetical or non-existing components.

Neutrosophic numbers provide a framework for dealing with incomplete, imprecise, or uncertain information in a more flexible manner than classical numbers. They have applications in various fields such as artificial intelligence, decision-making, information fusion, and other areas where uncertainty needs to be explicitly represented and manipulated in mathematical models.

3.2 | Neutrosophic Operations

One notes, with respect to the sets A and B over the universe U, $x = x\ (\mathrm{T_1}, \mathrm{I_1}, \mathrm{F_1}) \in A$ and $x = x\ (\mathrm{T_2}, \mathrm{I_2}, \mathrm{F_2}) \in B$, by mentioning x's Neutrosophic membership, indeterminacy, and non-membership respectively appurtenance. And, similarly, $y = y\ (T\,', I\,', F\,') \in B$. If, after calculations, in the below operations one obtains values < 0 or > 1, then one replaces them with -0 or 1^+ respectively. There are operations like Complement of A, Intersection, Union, Difference, Cartesian product etc.

3.3 | Neutrosophic Matrices and Linear Algebra

Neutrosophic linear algebra is an extension of classical linear algebra that incorporates the concept of neutrosophy. Neutrosophy is a branch of philosophy that deals with the study and formalization of the origin, nature, and scope of neutralities, as well as the interactions between different types of neutralities. In the context of Linear algebra, Neutrosophic Linear algebra extends traditional Linear algebra by allowing the elements of matrices, vectors, and other algebraic structures to be indeterminate, ambiguous, or neutral. In traditional Linear algebra, elements are typically real or complex numbers and operations are well defined. However, in Neutrosophic linear algebra, elements can belong to the set of real or complex numbers, but they can also be indeterminate, meaning that their values are uncertain, ambiguous, or neutral. Neutrosophic Linear algebra introduces the concept of Neutrosophic numbers, which are triples (a, b, c), where 'a' is the determinate component, 'b' is the indeterminate component, and 'c' is the neutral or imaginary component.

3.4 | Neutrosophic Set Theory

3.4.1 | Neutrosophic set operations

Let M and N be two Neutrosophic sets. One can say, by language abuse, that any element Neutrosophically belongs to any set, due to the percentage of truth/indeterminacy/falsity, which varies between 0 and 100. For example: x (50, 20, 30) ∈ M (Which means, with a probability of 50% x is in M, with a probability of 30% x is not in M, and the rest is undecidable) y (O, O, I 00) ∈ M (Which normally means y is not for sure in M), or z (O, 100, 0) ∈M (Which means one doesn't know absolutely anything about z's affiliation with M).

Definition 1. A Single-Valued Neutrosophic Set (SVNS) A in the universe U is of the form:

$$A = \{(x, T(x), I(x), F(x)): x \in U, T(x), I(x), F(x) \in [0,1], 0 \le T(x)+I(x)+F(x) \le 3\}. \quad (1)$$

In *Eq. (1)*, $T(x)$, $I(x)$, $F(x)$ are the degrees of truth (Or membership), indeterminacy (Or neutrality) and falsity (Or non-membership) of x in A respectively, called the Neutrosophic components of x. For simplicity, we write A. Indeterminacy is defined to be in general everything that exists between the opposites of truth and falsity.

Definition 2. Let (t_1, i_1, f_1), (t_2, i_2, f_2) be in A and let k be a positive number. Then

I. The sum $(t_1, i_1, f_1) + (t_2, i_2, f_2) = (t_1 + t_2, i_1+i_2, f_1+ f_2)$.

II. The scalar product $k(t_1, i_1, f_1) = (kt_1, ki_1, kf_1)$.

Definition 3. Let A be a SVNS and let (t_1, i_1, f_1), (t_2, i_2, f_2), …., (t_k, i_k, f_k) be a finite number of elements of A. Assume that (t_i, i_i, f_i) appears n_i times in an application, i = 1,2,…, k. Set $n = n_1+n_2+\ldots+n_k$. Then the mean value of all these elements of A is defined to be the element of A.

$$(t_m, i_m, f_m) = 1/n\ [n_1(t_1, i_1, f_1 1)+n_2(t_2, i_2, f_2)+\ldots.+n_k(t_k, i_k, f_k)].$$

3.4.2 | Neutrosophic subtraction algebra

A pair (A, −) where A is a nonempty set and '−' is a binary operation on A is called a subtraction algebra if

I. x − (y − x) = x.

II. x − (x − y) = y − (y − x).

III. (x − y) − z = (x − z) − y for all x, y, z ∈ A.

Axiom 3 permits us to omit parentheses in expressions of the form (x − y) − z. The subtraction determines an order relation on A: x ≤ y if and only if x − y = 0, where 0 = x − x is an element that does not depend on the choice of x ∈ A. The ordered set (A, ≤) is a semi-Boolean algebra, that is, it is a meet semi lattice with zero 0 in which every interval [0, x] is a Boolean algebra with respect to the induced order. Here x ∧ y = x − (x − y); the complement of an element y ∈ [0, x] is x − y; and if y, z ∈ [0, x], then

$$y \vee z = (y\ 0 \wedge z\ 0)\ 0 = x - ((x - y) \wedge (x - z)) = x - ((x - y) - ((x - y) - (x - z))).$$

In a subtraction algebra, the following hold:

I. $x - 0 = x$ and $0 - x = 0$.

II. $x - (x - y) \leq y$.

III. $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

IV. $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

V. $x - (x - (x - y)) = x - y$.

Definition 4. Let X be a nonempty set. A set $X(I) =< X \cup I >$ generated by X and I is called a Neutrosophic set. The elements of X(I) are of the form (x, yI) where x and y are elements of X. I is called an indeterminate and it has the property $I^n = I$ for all positive integer n.

Definition 5. Let (X, −) be any classical subtraction algebra and let $X(I) =< X \cup I >$ be a set generated by X and I. Consider the Neutrosophic algebraic structure (X(I), −N) where for all (a, bI),(c, dI) ∈ X(I), −N is defined by

$$(a, bI) -N\ (c, dI) = (a - c, (b - d)I), \text{ for all } a, b, c, d \in X.$$

We call (X(I), −N) a Neutrosophic subtraction algebra.

An element x ∈ X is represented by (x, 0) ∈X(I) and (0, 0) represents the constant element in X(I).

Neutrosophic logic

This is a generalization (For the case of null indeterminacy) of the fuzzy logic.

Neutrosophic logic is useful in the real-world systems for designing control logic, and may work in quantum mechanics.

If a proposition P is t% true, does not necessarily mean it is 100-t% false as in fuzzy logic. There should also be a percent of indeterminacy on the values of P.

A better approach of the logical value of P is f% false, i% indeterminate, and t% true, where t+i+f= 100 a.'1d t, i, f ∈ [0,100], called Neutrosophic logical value of P, and noted by n(P) = (t, i, f). Neutrosophic logic means the study of Neutrosophic logical values of the propositions. There exist, for each individual event, PRO parameters, CONTRA parameters, and NEUTER parameters which influence the above values. Indeterminacy results from any hazard, which may occur, from unknown parameters, or from new arising conditions. This resulted from practice.

3.5 | Neutrosophic Probability Theory and Statistics

3.5.1 | Definition

Neutrosophic Probability studies the chance that a particular event E will occur, where that chance is represented by three coordinates (Variables): t% true, i% indeterminate, and f% false, with t+i+f = 100 and f, i, t ∈ [0,100]. Neutrosophic Statistics is the analysis of such events.

In Imprecise Probability (IP), the probability of an event A.

$$IP(A) = (a, b) \subseteq [0,1],$$

is an interval included into [0,1], not a crisp number.

The Neutrosophic Probability that an event A occurs is

$$NP(A) = \big(ch(A), ch(neutA), ch(antiA)\big) = (T, I, F),$$

but sometimes instead of "Neuta" we say "Indeterminacy related to A" and we denote it by "$indeterm_A$"; also we denote "Antia" by $\bar{A}$;

where T, I, F are standard or nonstandard subsets of the nonstandard unitary interval, and the T is the chance that A occurs, denoted $\mathrm{ch}(A)$; I is the indeterminate chance related to A, $\mathrm{ch}(\mathrm{indeterm}_A)$; and F is the chance that A does not occur, $\mathrm{ch}(\overline{A})$.

So, NP is a generalization of the IP as well. Therefore, using other notations we have

$$\mathrm{NP}(A) = \left(\mathrm{ch}(A), \mathrm{ch}(\mathrm{indeterm}_A), \mathrm{ch}(\overline{A})\right).$$

3.5.2 | Neutrosophic probability space

The universal set, endowed with a Neutrosophic probability defined for each of its subset, forms a Neutrosophic probability space.

3.6 | Neutrosophic Group, Pseudo Group and Semigroup

3.6.1 | Neutrosophic group

Definition 6. Let (G, *) be any group, the Neutrosophic group is generated by I and G under * denoted by $N(G) = \{\langle G \cup I\rangle, *\}$.

Example 1. Let Z_7 = {0, 1, 2,…, 6} be a group under addition modulo 7. N(G) = $\{\langle Z_7 \cup I\rangle$, '+' modulo 7} is a Neutrosophic group which is in fact a group. For N(G) = {a + bI / a, b $\in Z_7\}$ is a group under '+' modulo 7. Thus this Neutrosophic group is also a group.

Example 2. Consider the set G= $Z_5 \backslash \{0\}$, G is a group under multiplication modulo 5. N(G) = $\{\langle G \cup I\rangle$, under the binary operation, multiplication modulo 5}. N(G) is called the Neutrosophic group generated by $G \cup I$. Clearly N(G) is not a group, for $I^2 = I$ and I is not the identity but only an indeterminate, but N(G) is defined as the Neutrosophic group.

Theorem 1. Let (G, *) be a group, N(G) = $\{\langle G \cup I\rangle, *\}$ be the Neutrosophic group.

I. N(G) in general is not a group.

II. N(G) always contains a group.

Proof: To prove N(G) in general is not a group it is sufficient we give an example; consider $\langle Z_5 \setminus \{0\} \cup I\rangle = G = \{1, 2, 4, 3, I, 2I, 3I, 4I\}$; G is not a group under multiplication modulo 5. In fact {1, 2, 3, 4} is a group under multiplication modulo 5.

N(G) the Neutrosophic group will always contain a group because we generate the Neutrosophic group N(G) using the group G and I. So G ≠ ⊂N(G); hence N(G) will always contain a group.

Definition 7. Let N(G) be a Neutrosophic group. The number of distinct elements in N(G) is called the order of N(G). If the number of elements in N(G) is finite we call N(G) a finite Neutrosophic group; otherwise we call N(G) an infinite Neutrosophic group, we denote the order of N(G) by o(N(G)) or |N(G)|.

Definition 8. Let N(G) be a Neutrosophic group. An element x ∈ N(G) is said to be a Neutrosophic element if there exists a positive integer n such that xn = I, if for any y a Neutrosophic element no such n exists then we call y to be a Neutrosophic free element.

3.6.2 | Neutrosophic pseudo group

Definition 9. Let Gbe a set equipped with a binary, $G: G \times G \to G$. A Neutrosophicpseudogroup $(G, .)$is defined when it satisfies the following properties:

I. Neutrosophic closure property: For all elements $a, b \in G$, the result of the operation $a \,.\, b$ is a Neutrosophic element in G.

II. Neutrosophic associativity property: For all elements $a, b, c \in G$, the operation satisfies a form of associativity that accounts for indeterminacy. That is, $(a . b) . c$ and $a . (b . c)$ are indeterminately associative.

III. Neutrosophic identity: There exists an element $e \in G$ (The Neutrosophic identity element) such that for all elements $a \in G$, the operation satisfies $a . e \approx e . a \approx a$, where ≈ denotes approximate equality under the Neutrosophic framework.

IV. Neutrosophic inverses: For each element $a \in G$, there exists an element $b \in G$ (The Neutrosophic inverse of a) such that $a . b \approx b . a \approx e$, where ≈ denotes approximate equality under the Neutrosophic framework.

V. Neutrosophic elements: Each element $a \in G$ is represented as a Neutrosophicnumber $a = (T_a, I_a, F_a)$, where T_a, I_a and F_a are the truth, indeterminacy, and falsehood components of a, respectively.

Definition 10. Let $N(G) = \langle G \cup I \rangle$ be a neutrosophic group generated by G and I. A proper subset P(G) is said to be a Neutrosophic subgroup if P(G) is a Neutrosophic group i.e. P(G) must contain a (Sub) group.

Example 3. Let $N(Z_2) = \langle Z_2 \cup I \rangle$ be a Neutrosophic group under addition. $N(Z_2) = \{0, 1, I, 1 + I\}$. Now we see {0, I} is a group under + in fact a NEUTROSOPHIC group {0, 1 + I} is a group under '+' but we call {0, I} or {0, 1 + I} only as pseudo Neutrosophic groups for they do not have a proper subset which is a group. So {0, I} and {0, 1 + I} will be only called as pseudo Neutrosophic groups (Subgroups).

We can thus define a pseudo Neutrosophic group as a Neutrosophic group, which does not contain a proper subset which is a group. Pseudo Neutrosophic subgroups can be found as a substructure of Neutrosophic groups. Thus a pseudo Neutrosophic group though has a group structure is not a Neutrosophic group and a Neutrosophic group cannot be a pseudo Neutrosophic group. Both concepts are different.

3.6.3 | Neutrosophic semigroup

Definition 11. Let S be a semigroup, the semigroup generated by S and I i.e. $S \cup I$ denoted by $\langle S \cup I \rangle$ is defined to be a Neutrosophic semigroup.

Note: All Neutrosophic semigroups contain a proper subset, which is a semigroup.

Definition 12. Let N(S) be a Neutrosophic semigroup. The number of distinct elements in N(S) is called the order of N(S), denoted by o(N(S)).

Definition 13. Let N(S) be a Neutrosophic semigroup. A proper subset P of N(S) is said to be a Neutrosophic subsemigroup, if P is a Neutrosophic semigroup under the operations of N (S). A Neutrosophic semigroup N(S) is said to have a subsemigroup if N(S) has a proper subset, which is a semigroup under the operations of N(S).

Theorem 2. Let N(S) be a Neutrosophic semigroup. Suppose P_1, P_2 be any two Neutrosophic subsemigroups of N(S) then $P_1 \cup P_2$ (i.e. the union) the union of two Neutrosophic subsemigroups in general need not be a Neutrosophic subsemigroup.

3.7 | Neutrosophic Field

Definition 14. Let K be the field of reals. We call the field generated by $K \cup I$ to be the Neutrosophic field for it involves the indeterminacy factor in it. We define $I^2 = I$, $I + I = 2I$ i.e., $I + I + ... + I = nI$, and if $k \in K$ then $k.I = kI$, $0I = 0$. We denote the Neutrosophic field by $K(I)$ which is generated by $K \cup I$ that is $K(I) = \langle K \cup I \rangle$. $\langle K \cup I \rangle$ denotes the field generated by K and I.

Example 4. Let R be the field of reals. The Neutrosophic field of reals is generated by R and I denoted by $\langle R \cup I \rangle$ i.e. $R(I)$ clearly $R \subset \langle R \cup I \rangle$.

3.8 | Neutrosophic Ring

Definition 15. Let R be any ring. The Neutrosophic ring $\langle R \cup I \rangle$ is also a ring generated by R and I under the operations of R.

Example 5. Let Z be the ring of integers; $\langle Z \cup I\rangle = \{a + bI | a, b \in Z\}$. $\langle Z \cup I\rangle$ is a ring called the Neutrosophic ring of integers. Also $Z \neq \subseteq \langle Z \cup I\rangle$.

4 | Application of Neutrosophic Algebra in Decision Making Processes

4.1 | Decision Making

DM is a fundamental process in a great spectrum of human activities and many books have been written about it, helping decision makers to make smarter choices easier and quicker. Frequently in real life situations, however, DM takes place under fuzzy conditions, since the corresponding goals and/or the existing constraints are not clearly defined. Several methods have also been proposed for successful DM in such cases.

4.1.1 | Multi-criteria decision making

Multiple-Criteria Decision Making (MCDM) or Multiple-Criteria Decision Analysis (MCDA) is a sub-discipline of operations research[1] that explicitly evaluates multiple conflicting criteria[2] in DM[3] (Both in daily life and in settings such as business, government and medicine). It is also known as multiple attribute utility theory, multiple attribute value theory, multiple attribute preference theory, and multi-objective decision analysis.It is particularly useful in complex decision-making scenarios where various criteria must be considered simultaneously. Common methods of MCDM includes:

I. Analytic Hierarchy Process (AHP): Developed by Thomas Saaty, AHP involves decomposing a complex decision problem into a hierarchy of simpler sub-problems. Each level of the hierarchy represents different criteria or sub-criteria, and pairwise comparisons are used to establish priorities.

II. Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS): TOPSIS ranks alternatives based on their distance from an ideal solution (The best possible scenario) and a nadir solution (The worst possible scenario). The alternative closest to the ideal solution and farthest from the nadir solution is preferred.

III. Elimination and Choice Expressing Reality (ELECTRE): ELECTRE is a family of MCDM methods that uses pairwise comparisons to eliminate less favorable alternatives. It is particularly useful for dealing with qualitative criteria.

IV. Multi-Attribute Utility Theory (MAUT): MAUT involves assigning utility values to different levels of criteria and then calculating a total utility score for each alternative. The alternative with the highest utility score is chosen

V. Simple Additive Weighting (SAW): Also known as the weighted sum method, SAW involves assigning weights to criteria, multiplying the scores of alternatives by these weights, and summing the results to obtain a final score for each alternative.

VI. VlseKriterijumskaOptimizacija I KompromisnoResenje (VIKOR): VIKOR focuses on ranking and selecting from a set of alternatives, and determining compromise solutions, with respect to conflicting criteria.

4.2 | Application of Neutrosophic Algebra

Neutrosophic algebra has contributed both theoretically and practically to mathematics and science at large. For now, most applications of Neutrosophic algebra remains theoretical.

4.2.1 | Theoretical examples

I. Multi-refined Neutrosophic set has also been utilized in sentiment analysis for Data Mining. The model consisted of two positive (T), two negative (F), and three indeterminate (I) membership functions. Their

[1] https://en.wikipedia.org/wiki/Operations_research

[2] https://en.wiktionary.org/wiki/criterion

[3] https://en.wikipedia.org/wiki/Decision_making

model showed outperforming results when analysing tweets on ten different topics related to the Indian scenario and other international scenarios.

II. A utilized single-valued and interval-valued Neutrosophic graphs in Blockchain and bitcoin application was proposed. They also listed the advantages and limitations of Blockchain graphs. Another model was also proposed and their main focus was to select the most appropriate Blockchain model for providing a secure and trustworthy healthcare Blockchain solution. Their proposed model was well-presented yet had a set of limitations, including its generalization.

III. Neutrosophic PROMETHEE method in one-to-one marketing. They proved the necessity of analysing different aspects of potential buyers, including their emotional and physiological states. One of the main obstacles of their model was the collection, and governance of the customers' emotions related data, especially under the existence of the data privacy rules exist in many companies. Consequently, collecting the proper data needed time and cost.

IV. From 2020 until publishing this research, the whole world is struggling with different generations of Coronavirus (i.e., COVID-19). Many researchers tried to contribute to developing models for dealing with COVID-19 using a different algorithm, and Neutrosophic logic was on top of these utilized algorithms. A frameworkwas developed that combined COVID-19's disruptive technologies for analyzing this pandemic virus. Their framework had many advantages, including restricting COVID-19's outbreaks and ensuring the healthcare team's safety. The power of their model was applying it in an empirical case study. Meanwhile, it would be great to be generalized on other use cases.While some other researchers, differentiated between COVID-19 and other four chest diseases that had some common symptoms. They utilized Neutrosophic logic for this purpose to diagnose COVID-19 using only the CT scan and the primary symptoms. They also studied the effect of the Internet of Things (IOT) in helping the medical staff monitor the spread of COVID-19. Their proposed model achieved 98% detection accuracy. Yet, it was recommended to update their study by including other COVID-19's symptoms added by the World Health Organization (WHO) related to the virus's evolution.

V. Some researchers combined Neutrosophic techniques with AHP method. Their main goal was to support enterprise decision-making in the Internet of Things (IoT) era. Their proposed algorithm was applied in different enterprises, including Smart Village in Egypt and Smart City in U.K. and China. Meanwhile, involving more companies in their model validation would enrich its results. Another set of researchers designed an interval complex Neutrosophic set and listed its characteristics. To prove their model's practicality, they applied it in supplier selection related to a transportation company. Meanwhile, it still needed to be applied to more real-life datasets to prove its robustness.Then, researchers developed a supply chain-related multi-criteria group Decision-Making Method. Their proposed model combined analytical network process method to VIKOR Method. Their model that was developed for a Neutrosophic environment that had incomplete information, utilized triangular membership function was applied in a real case study. It showed outperforming results. Its main limitation was the dependency on experts' opinions and it was hard to find those experts fulfilled the researchers' predefined requirements. Also, its dependency on a forecasting phase that needed to have large input data to have robust results. Meanwhile, it was recommended to generalize it to other applications. VIKOR Method was also utilized to guide the decision-making process in an uncertain environment in another research. For this purpose, they combined VIKOR method to the cubic Neutrosophic number. This combination was illustrated through an example but was recommended to be applied in real applications.

4.2.2 | Practical examples

Example 6. A soccer club wants to choose a new player among 6 candidates, say P_1, P_2, P_3, P_4, P_5 and P_6. The desired qualifications of the new player are to be fast, younger than 30 years, higher than 1.70 m and experienced. Assume that P_1, P_2, P_6 are the fast players, P_2, P_3, P_5,P_6 are the players being younger than 30 years, P_3, P_5 are the players with heights greater than 1.70 m and P_4 is the unique experienced player. Assume that the technical manager of the soccer club, being not sure about the qualitative grades assigned to each of the 6 candidate players, he decided to proceed by replacing them by Neutrosophic triplets, in the way that we have previously described. As a result, the tabular matrix of the DM process took the form of the following table below. Which is the best decision for the club in this case ?

Table 1. Neutrosophic decision matrix for 6 players across 4 criteria.

	e_1	e_2	e_3	e_4
P_1	(1, 0, 0)	(0, 0, 1)	(0, 0, 1)	(0.6, 0.3, 0.1)
P_2	(1, 0, 0)	(1, 0, 0)	(0, 0, 1)	(0.2, 0.2, 0.6)
P_3	(0.5, 0.4, 0.1)	(1, 0, 0)	(1, 0, 0)	(0.6, 0.2, 0.2)
P_4	(0.5, 0.2, 0.3)	(0, 0, 1)	(0, 0, 1)	(1, 0, 0)
P_5	(0.5, 0.1, 0.4)	(1, 0, 0)	(1, 0, 0)	(0.6, 0.3, 0.1)
P_6	(1, 0, 0)	(1, 0, 0)	(0, 0, 1)	(0.4, 0.4, 0.2)

Solution: The choice value of each player in this case is defined to be the mean value of the Neutrosophic triplets of the line of the table in which he belongs. Thus, by

$$(t_m, i_m, f_m) = 1/n\ [n_1(t_1, i_1, f_1 1)+n_2(t_2, i_2, f_2)+....+n_k(t_k, i_k, f_k)].$$

The choice value of P_1 is equal to

$$\frac{1}{4}[(1, 0, 0)+2(0, 0, 1)+(0.6, 0.3, 0.1)] = \frac{1}{4}(1.6, 0.3, 2.1) = (0.4, 0.075, 0.525).$$

In the same way one finds that the choice values of P_2, P_3, P_4, P_5 and P_6 are (0.55, 0.005, 0.4), (0.775, 0.15, 0.075), (0.375, 0.05, 0.575), (0.775, 0.1, 0.125) and (0.6, 0.1, 0.3) respectively.

In this case, the club's technical manager could use either an optimistic criterion by choosing the player with the greatest truth degree, or a conservative criterion by choosing the player with the lower falsity degree. Consequently, using the optimistic criterion he must choose one of the players P_3 and P_5, whereas using the conservative criterion he must choose the player P_3. A combination of the two criteria leads to the final choice of player P_3. Observe, however, that, since the indeterminacy degree of P_3 is 0.15 and of P_5 is 0.1, there is a slightly greater doubt about the qualifications of P_3 with respect to the qualifications of P_5. In other words, the choice of P_3 is connected with a slightly greater risk. In final analysis, therefore, all the Neutrosophic components assigned to each player give useful information about his qualifications.

Example 7. The probability that candidate C will win an election is say 25% true (Percent of people voting for him), 35% false (Percent of people voting against him), and 40% indeterminate (Percent of people not coming to the ballot box, or giving a blank vote - not selecting anyone, or giving a negative vote-cutting all candidates on the list). Dialectic and dualism do not work in this case anymore.

Example 8. Another example, the probability that tomorrow it will rain is say 50% true according to meteorologists who have investigated the past years' weather, 30% false according to today's very sunny and droughty summer and 20% undecided (Indeterminate).

5 | Summary

This research involved a comprehensivereview of existing literature on Neutrosophic algebra and related areas,examining Neutrosophic algebra andits structure. Several journals, articles etc. was used to highlight the various branches of Neutrosophic algebra. The research reveals the significant role Neutrosophicalgebra is playing in solving diverse problems across fields like medicine, AI and machine learning, sports, blockchain, economics etc.

6 | Conclusion

The research suggests that Neutrosophic algebra will be greatly depended on to make real life day-to-day decisions and in creating decision-based models and algorithms. Moreover, will only expand across more fields as it accounts for uncertainty therefore proving a more accurate solution. Hence, it will soon added to upper education's mathematics curriculum and taught (Like Abstract and Linear algebra) as a branch of Mathematics.

Author Contributions

Joseph Oluwakorede Campbell was responsible for the theoretical formulation and literature examination. Adesina Adebisi Sunday directed the methodological design, analysis, and the preparation of the manuscript. Both authors evaluated and endorsed the final version.

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Conflicts of Interest

The authors state that there are no conflicts of interest.

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